# Neutron charge radius and the Dirac equation

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## Abstract

We consider the Dirac equation for a finite-size neutron in an external electric field. We explicitly incorporate Dirac-Pauli form factors into the Dirac equation. After a non-relativistic reduction, the Darwin-Foldy term is cancelled by a contribution from the Dirac form factor, so that the only coefficient of the external field charge density is $\frac{e}{6}r_{En}^2$, *i. e.* the root mean square radius associated with the electric Sachs form factor . Our result is similar to a recent result of Isgur, and reconciles two apparently conflicting viewpoints about the use of the Dirac equation for the description of nucleons.

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The description of a spin half particle ( e.g., neutron, proton) which has internal structure by means of the Dirac equation remains controversial. The seemingly innocuous assumption that the neutron behaves like a Dirac particle has led to quite a controversy over the meaning of the charge radius of the neutron. This quantity is measured in low energy neutron-atom scattering and thus it is linked to the electrical polarizability; sometimes both are extracted from the same experiment [1]. The disparate values [2,3] in the literature are cited as, on the one hand, consistent with chiral bag models [4] and, on the other hand, quite inconsistent with these same models of the substructure of the neutron [5]. This dispute is based on the denial or acceptance of the Foldy term [6] which arises in the use of a Dirac-Pauli equation for the interaction of the neutron with an external electromagnetic field [7]. The issue was recently revived by Isgur in the context of the valence quark model [8]. For the valence quark model, Isgur concludes that the Foldy term is cancelled by a contribution from the Dirac

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form factor. We will discuss here the form this cancellation takes when the Dirac equation is used in conjunction with electromagnetic form factors of the Dirac-Pauli form. That is, a Foldy-Wouthuysen type analysis of the Dirac equation yields a result similar to that of Isgur, but it makes no explicit reference to the nature of the internal structure of the neutron. The result follows only from the fact that the neutron is not a point particle and the assumption that the neutron is described by the Dirac equation with a form factor description of its non-zero size.

Consider the Dirac Hamiltonian for a neutron of mass m in an external electric field $\mathbf{E}$. We include form factors to describe the composite nature of the neutron. The natural form factors of the Dirac equation are $F_1(t)$ and $F_2(t)$, defined by the momentum space charged-matter four-current density $j_\kappa^+(x)$ in the presence of an electromagnetic field and for positive energy scattered particles:

$$e\langle\mathbf{p}'|j_\mu^+(0)|\mathbf{p}\rangle = e\bar{u}(\mathbf{p}')[F_1(t)\gamma_\mu + F_2(t)i\sigma_{\mu\nu}q^\nu/2m]u(\mathbf{p}) , \tag{1}$$

where $t = q^2 = (p - p')^2$ and $\bar{u}(\mathbf{p}')$ and $u(\mathbf{p})$ are free particle Dirac spinors. We employ the contemporary definitions of *dimensionless* form factors (see, for example, Ref. [9]), so that $F_1(0) = 1$ for a charged and $F_1(0) = 0$ for a neutral particle. Then $F_2(0) \equiv \kappa = \frac{1}{2}(g - 2)$ is the dimensionless, anomalous magnetic moment of a spin-$\frac{1}{2}$ particle [10]. The four momentum transferred from the photon to the spin $\frac{1}{2}$ particle is $q = p - p'$. In the Breit frame, necessary for assigning a coordinate space meaning to the nucleon electromagnetic form factors, $\mathbf{p} + \mathbf{p}' = 0$ so that $q_0 = 0$, and $t = (p - p')^2 = -\mathbf{q}^2$. The electric and magnetic Sachs form factors are defined by

$$G_E(t) = F_1(t) + \frac{t}{4m^2}F_2(t)$$
$$G_M(t) = F_1(t) + F_2(t) \tag{2}$$

In the Breit frame, the Sachs form factors have simple interpretations as the spatial Fourier transforms of the nucleon's charge and magnetization distributions [11]. We need only

$$\rho_N(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^3 \int d^3\mathbf{q}\, e^{+i\mathbf{q}\cdot\mathbf{r}} G_E(-\mathbf{q}^2) , \tag{3}$$

such that the normalization $\int d^3\mathbf{r}\,\rho_N(\mathbf{r})$ is one for the proton and zero for the neutron.

We will concentrate on the small momentum transfer aspects of this problem, so write the usual expansions of $F_1$ and $F_2$:

$$F_i(t) = F_i(0) + tF_i'(0) + \cdots \quad , i = 1, 2 , \tag{4}$$

so that

$$G_E'(t) = F_1'(t) + \frac{F_2(t)}{4m^2} + \frac{t}{4m^2}F_2'(t) , \tag{5}$$

and

2

$$G_E'(0) = F_1'(0) + \frac{F_2(0)}{4m^2} = F_1'(0) + \frac{\kappa}{4m^2} \, , \tag{6}$$

where $\kappa$ *is* the (dimensionless) magnetic moment of the neutron. However, one defines the root mean square radius of the charge distribution of the neutron by means of the inverse Fourier transform

$$G_E(-\mathbf{q}^2) = \int d^3 r e^{-i\mathbf{q}\cdot\mathbf{r}} \rho(\mathbf{r}) = -\frac{1}{6} r_{En}^2 \mathbf{q}^2 + \cdots \, , \tag{7}$$

with the corresponding definition $G_E(-\mathbf{q}^2) = 1 - \frac{1}{6} r_{Ep}^2 \mathbf{q}^2 + \cdots$ for the proton. Since $G_E'(t = 0) = -\frac{\partial G_E(\mathbf{q}^2=0)}{\partial \mathbf{q}^2}$, equation (6) becomes

$$F_1'(0) \equiv \frac{\partial F_1(\mathbf{q}^2 = 0)}{\partial \mathbf{q}^2} = -\frac{1}{6} r_{En}^2 + \frac{\kappa}{4m^2} \, . \tag{8}$$

After this discussion of the electromagnetic form factors we are ready to write down the Dirac Hamiltonian operator as:

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m + i\frac{eF_2}{2m}\beta\boldsymbol{\alpha} \cdot \mathbf{E} + eF_1 V \, , \tag{9}$$

where $F_1$ and $F_2$ are, respectively, the neutron Dirac and Pauli form factors, and $V$ is the potential associated with $\mathbf{E}$. Note that the use of Breit variables then leads in a natural way to using the Coulomb gauge for $V(\mathbf{q})$. ¿From equations (5) and (6) it should be apparent that only $F_2(0) = \kappa$ plays a role in the charge radius of the neutron, so we rewrite (9) as

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m + i\frac{e\kappa}{2m}\beta\boldsymbol{\alpha} \cdot \mathbf{E} + eF_1 V \tag{10}$$

The latter (10) is then a simple extension of the model used in our discussion of the neutron polarizability [12,13]. Replacement of the function $F_2$ by its limit does not lead to a loss of generality but does simplify the non-relativistic reduction of the Dirac equation $H\psi = E\psi$, to which we now turn. We work in coordinate space where

$$\tilde{V}(\mathbf{r}) \equiv eF_1 V = e\left(\frac{1}{2\pi}\right)^3 \int d^3 \mathbf{q} \, e^{i\mathbf{q}\cdot\mathbf{r}} F_1(-\mathbf{q}^2) V(\mathbf{q}) \, , \tag{11}$$

and we retain the symbol $\mathbf{p}$ for the momentum operator $-i\boldsymbol{\nabla}$ for clarity. We begin the non-relativistic limit by squaring $H$ of (10) to get [14]

$$\{(E - \tilde{V})^2 - \mathbf{p}^2 - m^2 - \frac{e^2\kappa^2\mathbf{E}^2}{4m^2} + i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\tilde{V} + \frac{e\kappa}{2m}\beta(\boldsymbol{\nabla} \cdot \mathbf{E}) + 2\frac{e}{2m}\boldsymbol{\kappa} \cdot (\mathbf{E} \times \mathbf{p})\}\psi = 0 \, . \tag{12}$$

This equation is exact. Now we set $E \sim m$, $\beta \sim 1$, $\tilde{V} \ll m$, and employ the approximation $\boldsymbol{\alpha} \sim \frac{\mathbf{P}}{2m}$ discussed in equation (12.7) of Bethe and Salpeter's treatise [14]. We then get

3

$$\{\frac{\mathbf{p}^2}{2m} - \frac{e\mathbf{p}}{2m^2} \cdot (\mathbf{E} \times \boldsymbol{\kappa}) + \frac{e^2\kappa^2\mathbf{E}^2}{8m^3} - \frac{e\kappa}{4m^2}(\boldsymbol{\nabla} \cdot \mathbf{E})$$
$$+\tilde{V} + \frac{\boldsymbol{\nabla}^2}{8m^2}\tilde{V}\}\psi = E\psi \ . \tag{13}$$

Here we have separated the nonrelativistic terms new to this Hamiltonian ( *i.e.*, those with $\tilde{V}$ which vanish for a point Dirac particle) from those already displayed in equation (4) of Ref. [12] obtained by a Foldy-Wouthuysen transformation. We have checked that the systematic Foldy-Wouthuysen coordinate space procedure, which is more tedious and requires, in our case, switching back and forth between the coordinate and momentum representations of the Dirac equation, yields the same result (13). After completion of this work, we learned of the Hamiltonian obtained by McVoy and van Hove [15] using the Foldy-Wouthuysen procedure and checked that it is equivalent to equation (13).

In order to exhibit explicitly the $r_{En}^2$ in the non-relativistic equation (13) we must return to momentum space using the definition $\tilde{V}(\mathbf{q}) = eF_1(-\mathbf{q}^2)V(\mathbf{q})$ implicit in (11). Now we use the low $\mathbf{q}^2$ expansion of $F_1$ (see (4)) and write

$$\tilde{V}(\mathbf{q}) = e \int d^3\mathbf{q}\, e^{i\mathbf{q}\cdot\mathbf{r}} F_1(-\mathbf{q}^2)V(\mathbf{q}) = e \int d^3\mathbf{q}\, e^{i\mathbf{q}\cdot\mathbf{r}}[0 + \mathbf{q}^2 F_1'(0) + \cdots]V(\mathbf{q})$$
$$\approx -eF_1'(0)\boldsymbol{\nabla}^2 V(\mathbf{r}) = eF_1'(0)\boldsymbol{\nabla} \cdot \mathbf{E} = -e\left(\frac{1}{6}r_{En}^2 - \frac{\kappa}{4m^2}\right)\boldsymbol{\nabla} \cdot \mathbf{E} \ , \tag{14}$$

where we have used (8) and remind the reader that $\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}V(\mathbf{r})$.

By substituting (14) into (13), we observe that the Foldy term (the term $\frac{e\kappa}{4m^2}$) in (13) is now cancelled by a contribution from the *Dirac* form factor, leaving $\frac{e}{6}r_{En}^2$ as the only coefficient of the external field charge density. This is similar to Isgur's result [8]. Our result, however, is independent of any definite quark substructure of the neutron. The neutron just has to have a form factor. Furthermore, this analysis is in keeping with the philosophy that the basic equations for the neutron should be expressed in terms of a Dirac Hamiltonian, while the physical picture emerges from a nonrelativistic reduction which only contains $r_{En}^2$. In our analysis, we express the basic interaction in terms of a Dirac form factor $F_i$, but acknowledge that the physics lies in Sachs form factors $G_E$ and $G_M$.

In order to avoid a misunderstanding of our result, we note that it is the $\kappa$ term buried in the Sach's form factor $G_E$ (2) which contributes the most to the neutron charge radius, even after the cancellation of the Foldy term. The dominance of the $\kappa$ term in the neutron charge radius has been noted before by Friar [16].

It is of some interest to note that the left hand side of (13) also contains a term $\frac{\boldsymbol{\nabla}^2}{8m^2}\tilde{V}$, which has the form of a Darwin term, but this time associated with the *neutron* charge distribution rather than that of the external field. In particular, for $V = -e/r$ this term is simply

$$\frac{\boldsymbol{\nabla}^2}{8m^2}\tilde{V}(\mathbf{r}) = \frac{+e}{8m^2}\rho_{En}^1(\mathbf{r}) \ , \tag{15}$$

where

4

$$\rho_{En}^1(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^3 \int d^3\mathbf{q}\, e^{i\mathbf{q}\cdot\mathbf{r}} F_1(-\mathbf{q}^2) \,. \tag{16}$$

Finally, let us note that our analysis and conclusion are in complete agreement with the analysis of low-energy Compton scattering from the nucleon by L'vov [17]. In that work Dirac form factors appear in an effective Lagrangian for the $\gamma n$ interaction, but the relation (8) is (in effect) used in order to exhibit the low energy Compton amplitude expressed in terms of physical observables, as required by low energy theorems [18,19]. Our result also reconciles the two apparently conflicting viewpoints [4,5] about the use of the Dirac equation for the description of nucleons.

## ACKNOWLEDGEMENTS

# REFERENCES

[1] L. Koester, W. Waschkowski, L. V. Mitsyna, G. S. Samosvat, P. Prokofjevs, and J. Tambergs, "Neutron-electron scattering length and electric polarizability of the neutron derived from cross sections of bismuth and of lead and its isotopes", Phys. Rev. C **51**, 3363(1995).

[2] H. Leeb and C. Teichtmeister, "Charge radius of the neutron: Discussion of the differences between experimental values", Phys. Rev. C **48**, 1719(1993).

[3] S. Kopecky, P. Riehs, J. A. Harvey, and N. W. Hill, "New measurement of the charge radius of the neutron", Phys. Rev. Lett. **74**, 2427(1995).

[4] J. Byrne, "The mean square charge radius of the neutron", Neutron News, **5**, no. 4, 15(1994).

[5] Yu. A. Alexandrov, "What is the mean square charge radius of the neutron actually equal to?", Neutron News, **5**, no. 1, 20(1994); "On the mean square intrinsic charge radius of the neutron", Neutron News, **5**, no. 4, 17(1994) (reply to [4]); and "The sign and value of the neutron mean squared intrinsic charge radius", Phys. Part. and Nuc. **30**, 29(1999).

[6] L. L. Foldy, "Neutron-electron interaction", Rev. Mod. Phys. **30**, 471(1958).

[7] L. Ya. Glozman and D. O. Riska, "Pionic Fluctuations of constituent quarks and the neutron charge radius", `hep-ph/9812224`.

[8] N. Isgur, "Interpreting the neutron's electric form factor: rest frame charge distribution or Foldy term?", `hep-ph/9812243`.

[9] M. D. Scadron, *Advanced Quantum Theory and Its Applications Through Feynman Diagrams* (Springer-Verlag,Berlin,1990).

[10] We use Gaussian units so that the fine structure constant $\alpha = e^2/\hbar c \approx 1/137$. The unit of charge is $e > 0$, so that the electron has charge $-e$. The magnetic moment is the true *dimensionless* magnetic moment $\mu/\mu_N = (Z + \kappa) = \kappa$ where $Z = 0$ and $\kappa = -1.91$ for the neutron. The nuclear Bohr magneton $\mu_N = (e\hbar/2mc) \sim 5 \times 10^{-14} \mathrm{MeVT}^{-1}$ is a positive number. We set $\hbar = c = 1$ in the formulas of the text and now have $\boldsymbol{\kappa} = \kappa\boldsymbol{\sigma}$.

[11] R. G. Sachs, "High-energy behavior of nucleon electromagnetic form factors", Phys. Rev. **126**, 2256(1962). These form factors were known and correctly analyzed in the early review by D. R. Yennie, M. M Lévy, and D. G. Ravenhall "Electromagnetic Structure of Nucleons". Rev. Mod. Phys. **29**, 144(1957), but their interpretation as charge and magnetization distributions was established in Appendix II of the paper by Sachs.

[12] M. Bawin and S. A. Coon, "Dirac-Foldy term and the electromagnetic polarizability of the neutron", Phys. Rev. C **55**, 419(1997).

[13] The Dirac Hamiltonian of [12] had a different sign for $\mu = \kappa$ (mistakenly forced by us to agree with discussions of the Aharonov-Casher effect in the literature). Imposition of isospin symmetry in the form factors of the nucleon, as is manifest in the present paper, yields equation (10), the correct Dirac Hamiltonian: E. Corinaldesi and F. Strocchi, *Relativistic Wave Mechanics* (North-Holland,Amsterdam,1963) pp. 174-187; S.R. de Groot and L.G. Suttorp, *Foundations of Electrodynamics*, (North-Holland,Amsterdam,1972) p. 425. To make equations (3)-(8) of [12] consistent with (10) *et seq.* of the present

paper, change $\mu$ to $-\mu$ (so that $\kappa = -1.91$ for the neutron). This change does not alter the results of [12].

[14] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One-and two-electron atoms* (Springer-Verlag,Berlin,1957) pp. 47-58.

[15] K. W. McVoy and L. van Hove, "Inelastic electron-nucleus scattering and nucleon-nucleon correlations", Phys. Rev. **125** 1034(1962).

[16] J. L. Friar, "Meson exchange currents and relativistic effects", in *Electron and Pion Interactions with Nuclei at Intermediate Energies*, ed. W. Bertozzi, S. Costa, and C. Schaerf (Harwood, New York, 1980) pp. 175-201.

[17] A. I. L'vov, "Theoretical aspects of the polarizability of the nucleon", Int. J. Mod. Phys. A **8**, 5267(1993).

[18] V. M. Shekhter, "Compton effect on a nucleon at low energy", Sov. J. Nuc. Phy. **7**, 756(1968). This paper shows in a very clear fashion that $r_{En}^2$ and $r_{Ep}^2$ are *not* a part of the nucleon polarizability defined and measured in Compton scattering, in agreement with Refs. [12,17] and J. Bernabeu and R. Tarrach, "Long-range potentials and the electromagnetic polarizabilities", Ann. Phys. (N.Y.) **102**, 323(1976), but in disagreement with much of the literature which introduces non-relativistic concepts into the problem.

[19] M. Bawin and S. A. Coon, "Relativistic corrections to the electric polarizability of the neutron", Few-Body Systems Supp. n$^o$ 10, ed. B. Desplanques et al. (Springer-Verlag, Vienna, 1999), pp. 143-146, `nucl-th/9808015`.